Inverse Diffusion Curves using Shape Optimization

Supplementary Document

In this supplementary document, we first present (in §1) the detailed mathematical derivations of the formulas used in the paper. We also provide details (in §4) of our second-order finite element method (FEM) implementation for solving the Laplace's equation and Poisson's equation arose in the paper. While the derivation is mathematical, we will also provide intuitive interpretation of the formulas and definitions when possible.

1 Basics of Shape Optimization

To make this supplementary document self-contained, we first review the basics of the theory of Shape Optimization that our proposed method is built on. It also lays out a foundation for our derivations in §2. This section mostly follows the textbook [SZ92], where more detailed exposition of this topic is offered. The concept of shape derivative is concerned with the derivative value of a functional defined as a domain-related integral such as a domain integral and a boundary integral. It indicates how much the integral changes when the domain is deformed under a velocity field $v(\boldsymbol{x})$.

1.1 Domain-Related Integral

Let Ω denote a closed domain, and $L^1(\Omega)$ denote the L^1 functional space on Ω. We consider a domain integral:

$$J(\Omega) = \int_\Omega y(\boldsymbol{x};\Omega)d\Omega, \tag{1}$$

where the integrand $y \in L^1(\Omega)$ depends on the choice of the domain. One example is a function y depending on the total area of the domain Ω; another example is our diffusion curve residual (3) in the paper, as the color function $u(\boldsymbol{x})$ in the integrand depends on a specific domain Ω.

Another domain-related integral is the boundary integral:

$$L(\Gamma) = \int_\Gamma z(\boldsymbol{x};\Gamma)d\Gamma, \tag{2}$$

where $\Gamma = \partial\Omega$ is the boundary of Ω, and the integrand $z \in L^1(\Gamma)$ depends on the choice of the domain and thus its boundary. We will use the shape derivatives of both the domain integral (1) and boundary integral (2) later in this document.

1.2 Derivative of Domain Integral With Respect to Velocity Field

Now suppose there exists a velocity field $\boldsymbol{v}(\boldsymbol{x}), \forall \boldsymbol{x} \in \Omega$. Then, under the velocity field, we can define a time evolution of domain Ω_t starting from Ω_0 via

$$\Omega_t = \{\boldsymbol{x}(t)|\boldsymbol{x}(0)\in\Omega_0, \frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}(\boldsymbol{x}) \text{ in } (0,t)\} \text{ and } \Gamma_t = \partial\Omega_t.$$

Here the boundary of Ω_t is denoted as Γ_t. Intuitively, we consider Ω_0 consists of a set of particles. Those particles move along a pre-defined velocity field $v(\boldsymbol{x})$, starting from time 0. At time t, those particles from a deformed domain Ω_t. Mathematically, if $\boldsymbol{v}(\boldsymbol{x})$ is a Lipschitz continuous function (i.e., $\boldsymbol{v} \in C^{0,1}$), the Ω_t is always well-defined due to the Picard-Lindelöf theorem in the theory of ordinary differential equations [Arn83]. In addition, we define a family of transformations, $T_t : \Omega_0 \mapsto \Omega_t$ such that $T_t(\boldsymbol{x}_0), t = [0, T]$, is the trajectory of a particle starting from $\boldsymbol{x}_0 = \boldsymbol{x}(0)$ and moves under the velocity field $\boldsymbol{v}(\boldsymbol{x})$.

With the transformations T_t, we can rewrite the domain integral (1) over the domain Ω_t as an integral in the initial domain Ω_0. Using the classic domain transformation for integrals, we obtain

$$J(\Omega_t) = \int_{\Omega_t} y(\boldsymbol{x}; \Omega_t) d\Omega = \int_{\Omega_0} (y(\Omega_t) \circ T_t) \gamma(t) d\Omega, \tag{3}$$

where $\gamma(t) = \det(DT_t)$ is the determinant of the transformation gradient, and $(\cdot \circ \cdot)$ denotes a composition of two functions. This expression enables us to introduce the derivative of the domain integral (1).

The Fréchet derivative (also known as the *Eulerian Derivative*) of the functional $J(\Omega_t)$ at $t = 0$ is defined as

$$dJ(\Omega_0; \boldsymbol{v}) = \lim_{t \downarrow 0} \frac{J(\Omega_t) - J(\Omega_0)}{t}.$$

Here we use a slightly different notation from the ones in the paper: we explicitly write $\boldsymbol{v}$ as a parameter of dJ to emphasize the dependence of dJ on the velocity field. From the geometric intuition, this derivative is related to the boundary velocity $\boldsymbol{v}(\boldsymbol{x}), \forall \boldsymbol{x} \in \Gamma$. Known as the *Hadamard-Zeloésio Structure Theorem* [DZ11], this linear functional always exists when y, Ω and $\boldsymbol{v}$ are sufficiently regular. We now substitute (1) and (3) into the definition of the Fréchet derivative of the domain integral (1), and obtain

$$dJ(\Omega_0; \boldsymbol{v}) = \int_{\Omega_0} \left[\frac{d}{dt} (y(\Omega_t) \circ T_t) \gamma(t) + (y(\Omega_t) \circ T_t) \gamma'(t) \right] d\Omega \bigg|_{t=0}, \tag{4}$$

where $\gamma'(t)$ can be written as

$$\gamma'(t) = \mathbf{div}(\boldsymbol{v}(T_t(\boldsymbol{x}_0))) \gamma_t,$$

Noticing at $t = 0$, $\gamma_t = 1$ and $T_t(\boldsymbol{x}_0) = \boldsymbol{x}_0$, we further reduce the second term in the integral of (4) at $t = 0$ into

$$(y(\Omega_t) \circ T_t) \gamma'(t) \big|_{t=0} = y(\boldsymbol{x}_0; \Omega_0) \mathbf{div}(\boldsymbol{v}(\boldsymbol{x}_0)).$$

Next, the time derivative in the first term is defined as

$$\dot{y}(\Omega_0; \boldsymbol{v}) := \frac{d}{dt} (y(\Omega_t) \circ T_t) \bigg|_{t=0} = \lim_{t \downarrow 0} \frac{y(\Omega_t) \circ T_t - y(\Omega_0)}{t}.$$

Analogous to the concept of *material derivatives* in continuum mechanics [BW97], $\dot{y}(\Omega_0; \boldsymbol{v})$ is the material derivative of $y(\Omega_0)$ under the velocity field $\boldsymbol{v}$. Here again, we explicitly express $\dot{y}$ with a parameter $\boldsymbol{v}$ to emphasize its dependence on $\boldsymbol{v}$. Using this definition in (4), we obtain

$$\begin{aligned} dJ(\Omega_0; \boldsymbol{v}) &= \int_{\Omega_0} (\dot{y}(\Omega_0; \boldsymbol{v}) + y(\Omega_0) \mathbf{div}(\boldsymbol{v}))\, d\Omega \\ &= \int_{\Omega_0} (\dot{y}(\Omega_0; \boldsymbol{v}) - \nabla y(\Omega_0) \cdot \boldsymbol{v} + \mathbf{div}(y(\Omega_0) \boldsymbol{v}))\, d\Omega, \end{aligned} \tag{5}$$

where the second expression follows the integration by parts.

We now describe the concept of *shape derivative* of $y(\Omega_0)$ under the velocity field $\boldsymbol{v}$. It is defined as

$$y'(\Omega_0; \boldsymbol{v}) := \dot{y}(\Omega_0; \boldsymbol{v}) - \nabla y(\Omega_0) \cdot \boldsymbol{v}. \tag{6}$$

This concept is analogous to those in continuum mechanics: the material derivative is the time derivative value in material space. And the shape derivative is the derivative in deformed space; it measures the change rate of $y(\boldsymbol{x}_0)$ due to the boundary changes.

With the definition of shape derivatives, we rewrite (5) as

$$\begin{aligned} dJ(\Omega_0;\boldsymbol{v}) &= \int_{\Omega_0} \big(y'(\Omega_0;\boldsymbol{v}) + \mathbf{div}(y(\Omega_0)\boldsymbol{v}) \big)\, d\Omega \\ &= \int_{\Omega_0} y'(\Omega_0;\boldsymbol{v}) d\Omega + \int_{\Gamma_0} y(\Omega_0)\boldsymbol{v}\cdot\boldsymbol{n} d\Gamma. \end{aligned} \tag{7}$$

As a quick summary, we start from the transformation under a differential dynamical system specified by the velocity field $\boldsymbol{v}$. We rewrite the domain integral on Ω_t as a integral over the initial domain Ω_0. This allows us to define the Fréchet derivative of the domain integral. With the definition of material derivative and shape derivative, we simplify the Fréchet derivative of the domain integral into a form of (5). Next, we apply the same line of derivation for deriving the derivative of a boundary integral (2).

1.3 Derivative of Boundary Integral With Respect to Velocity Field

In the paper, when we describe the regularization of curve length (in §4.3 of the paper), we also use the Fréchet derivative of a boundary integral. We now present the related formulas, following the line of derivation in §1.2 above.

To begin with, we rewrite the boundary integral (2) over the boundary Γ_t as an integral over Γ_0, using the domain transformation T_t:

$$L(\Gamma_t) = \int_{\Gamma_t} z(\boldsymbol{x};\Gamma_t) d\Gamma = \int_{\Gamma_0} (z(\Gamma_t)\circ T_t)\omega(t) d\Gamma,$$

where $\omega(t)$ is related to the cofactor matrix of DT_t,

$$\omega(t) = |M(DT_t)\boldsymbol{n}(\boldsymbol{x}_0)| = \gamma(t)|(DT_t)^{-T}\boldsymbol{n}|.$$

Here D denote the spatial derivative operator, so DT_t is the Jacobian of the transformation T_t. $\boldsymbol{n}(\boldsymbol{x}_0)$ is the boundary normal direction at $\boldsymbol{x}_0$. Similar to the derivation in §1.2, we rewrite its Fréchet derivative as

$$dL(\Gamma_0;\boldsymbol{v}) = \int_{\Gamma_0} \left[\frac{d}{dt} (z(\Gamma_t)\circ T_t)\,\omega(t) + (z(\Gamma_t)\circ T_t)\omega'(t) \right] d\Gamma \Bigg|_{t=0}. \tag{8}$$

It can be checked that $\omega'(t)$ takes the following form:

$$\omega'(t) = \lim_{t\downarrow 0} \frac{\omega(t)-\omega(0)}{t} = \mathbf{div}(\boldsymbol{v}) - [(D\boldsymbol{v})\boldsymbol{n}]\cdot\boldsymbol{n}.$$

This expression is also the definition of *tangential divergence* of $\boldsymbol{v}$, denoted as $\mathbf{div}_\Gamma(\boldsymbol{v})$, since the divergence of $\boldsymbol{v}$ (the first term) is subtracted by the normal component of the divergence (the second term). Then, similar to the definition of $\dot{y}$ in §1.2 above, there is also a material derivative of $z(\Gamma_0)$ under the velocity $\boldsymbol{v}$:

$$\dot{z}(\Gamma_0;\boldsymbol{v}) = \frac{d}{dt}(z(\Gamma_t)\circ T_t)\Bigg|_{t=0} = \lim_{t\downarrow 0} \frac{z(\Gamma_t)\circ T_t - z(\Gamma_0)}{t}.$$

Th definitions of tangential divergence $\mathbf{div}_\Gamma(\boldsymbol{v})$ and material derivative $\dot{z}(\Gamma;\boldsymbol{v})$ further simplify the Fréchet derivative $dL(\Gamma;\boldsymbol{v})$ into

$$dL(\Gamma_0;\boldsymbol{v}) = \int_{\Gamma_0} \dot{z}(\Gamma_0;\boldsymbol{v}) d\Gamma + \int_{\Gamma_0} z(\Gamma_0)\mathbf{div}_{\Gamma_0}(\boldsymbol{v}) d\Gamma,$$

where $\mathbf{div}_{\Gamma_0}(\boldsymbol{v})$ is the tangential divergence of $\boldsymbol{v}$ on Γ_0, as introduced above.

Similar to the definition of y' in (6), we also have the shape derivative of $z'(\Gamma)$,

$$z'(\Gamma;\boldsymbol{v}) := \dot{z}(\Gamma;\boldsymbol{v}) - \nabla_\Gamma z(\Gamma)\cdot\boldsymbol{v}, \tag{9}$$

where we use the notation of *tangential gradient*, ∇_Γ, because z so far is defined only on the boundary Γ. In particular, let $\tilde{z}$ be an extension of z, $\tilde{z} \in C^2(U)$ and $\tilde{z}|_\Gamma = z$; U is an open neighborhood of Γ on $\mathbb{R}^2$. Then we can define the tangential gradient,

$$\nabla_\Gamma z = \nabla \tilde{z}|_\Gamma - \frac{\partial \tilde{z}}{\partial n}\boldsymbol{n}. \tag{10}$$

Lastly, it follows that

$$\begin{aligned} dL(\Gamma_0;\boldsymbol{v}) &= \int_{\Gamma_0} z'(\Gamma_0;\boldsymbol{v})d\Gamma + \int_{\Gamma_0} \left[\nabla_{\Gamma_0} z(\Gamma_0)\cdot\boldsymbol{v} + z(\Gamma_0)\mathbf{div}_{\Gamma_0}(\boldsymbol{v})\right]d\Gamma \\ &= \int_{\Gamma_0} z'(\Gamma_0;\boldsymbol{v})d\Gamma + \int_{\Gamma_0} \mathbf{div}_{\Gamma_0}\left(z(\Gamma_0)\boldsymbol{v}\right)d\Gamma \\ &= \int_{\Gamma_0} z'(\Gamma_0;\boldsymbol{v})d\Gamma + \int_{\Gamma_0} z(\Gamma_0)\kappa\boldsymbol{v}_n d\Gamma. \end{aligned} \tag{11}$$

where $\kappa = \mathbf{div}_{\Gamma_0}\boldsymbol{n}$ is the mean curvature on Γ_0, $\boldsymbol{n}$ is the normal direction on a boundary location, and $\boldsymbol{v}_n$ is the boundary normal velocity. As a simple example of using this formula, in §4.3 of the paper, we consider the boundary integral $\int_{\mathbb{B}} d\Gamma$, which is a special case for $z = 1$. Thus, $z' = 0$ in this case; and using the expression (11) above, we obtain Equation (14) of the paper.

1.4 Cost Functional for Curve Placement

After laying out the background about shape derivatives, we now introduce the cost functional that we propose in the paper. In the next subsection, we will derive formulas for minimizing this cost functional.

Consider the cost functional in §4.1 of the paper. We define the L_2 residual of diffusion curve approximation using the following domain integral:

$$J(\Omega_t) = \frac{1}{2}\int_{\Omega_t} (u(\boldsymbol{x};\Omega_t) - I(\boldsymbol{x}))^2 d\Omega, \tag{12}$$

where $u(\boldsymbol{x};\Omega_t) \in H^1(\Omega_t)$ is a weak solution of the Laplace's equation. According to (7), the Fréchet derivative of this functional is

$$dJ(\Omega_0;\boldsymbol{v}) = \int_{\Omega_0} (u(\Omega_0) - I)(u'(\Omega_0;\boldsymbol{v}) - I')d\Omega + \frac{1}{2}\int_{\Gamma_0} (u(\Omega_0) - I)^2 \boldsymbol{v}_n d\Gamma. \tag{13}$$

In the first term, the shape derivative of $I(\boldsymbol{x})$ vanishes, because

$$I'(\boldsymbol{x}) = \dot{I}(\boldsymbol{x}) - \nabla I(\boldsymbol{x})\cdot\boldsymbol{v} = \lim_{t\downarrow 0}\frac{I\circ T_t - I}{t} - \nabla I(\boldsymbol{x})\cdot\boldsymbol{v} = 0.$$

In the second term, we notice that $u(\boldsymbol{x};\Omega_0)|_{\Gamma_0} = I(\boldsymbol{x})$ because of the construction of the boundary condition of the Laplace's equation (i.e., the Equation (2) of the paper). Thus, the second term vanishes. And we obtain

$$dJ(\Omega_0;\boldsymbol{v}) = \int_{\Omega_0} (u(\Omega_0) - I)u'(\Omega_0;\boldsymbol{v})d\Omega. \tag{14}$$

1.5 Derivatives with PDE Constraints

We now outline the mathematical development for transforming the cost functional as linear forms of $\boldsymbol{v}_n$. As stated in the paper, given a color field $I(\boldsymbol{x}) : \mathbb{R}^2 \to \mathbb{R}$, we seek a set of diffusion curves satisfying

$$\begin{aligned} u(\boldsymbol{x};\Omega) &= I(\boldsymbol{x};\Omega), && \forall \boldsymbol{x}\in\Gamma_t, \\ \Delta u(\boldsymbol{x};\Omega) &= 0, && \text{otherwise}, \end{aligned} \tag{15}$$

to approximate the color filed I. Therefore, u is a harmonic function. Our central goal in this subsection is to express the Fréchet derivative (14) as linear form of $\boldsymbol{v}_n|_\Gamma$. We then extend the derivation to compute the derivatives of the cost functional introduced in §1.4.

We first derive the shape derivative of u satisfying the Dirichlet boundary value problem (15). From the Dirichlet boundary condition of (15), it follows that

$$\int_{\Gamma_t} u(\Omega_t)\phi \, d\Gamma_t = \int_{\Gamma_t} I(\Omega_t)\phi \, d\Gamma_t,$$

where Γ_t is the boundary of a domain Ω_t, and ϕ is an arbitrary test function in the Hilbert space $H_0^1(\Gamma_t)$. Both sides of this expression are boundary integrals over Γ_t. Taking the Fréchet derivative at $t=0$ on both sides (using (11)), we obtain

$$\begin{aligned}\int_{\Gamma_0} u'(\Omega_0;\boldsymbol{v})\big|_{\Gamma_0}\phi \, d\Gamma + \int_{\Gamma_0}\left[\frac{\partial}{\partial n}(u(\Omega_0)\phi) + \kappa u(\Omega_0)\phi\right]\boldsymbol{v}_n d\Gamma = \\ \int_{\Gamma_0} I'(\Omega_0;\boldsymbol{v})\big|_{\Gamma_0}\phi \, d\Gamma + \int_{\Gamma_0}\left[\frac{\partial}{\partial n}(I(\Omega_0)\phi) + \kappa I(\Omega_0)\phi\right]\boldsymbol{v}_n d\Gamma = \int_{\Gamma_0}\left[\frac{\partial}{\partial n}(I(\Omega_0)\phi) + \kappa I(\Omega_0)\phi\right]\boldsymbol{v}_n d\Gamma \end{aligned} \quad (16)$$

The last equality is due to the vanishing $I'(\Omega_0;\boldsymbol{v})$, because $I(\boldsymbol{X})$ is independent from the choice of a domain, as can be easily verified according to (6). Since ϕ can be any test function, if we further assume $\frac{\partial\phi}{\partial n}=0$ and notice that $u(\Omega;\boldsymbol{x}) = I(\Omega;\boldsymbol{x})$, $\forall \boldsymbol{x}\in\Gamma_0$, Equation (16) becomes

$$\int_{\Gamma_0} u'(\Omega_0;\boldsymbol{v})\big|_{\Gamma_0}\phi \, d\Gamma + \int_{\Gamma_0}\frac{\partial u(\Omega_0)}{\partial n}\phi \boldsymbol{v}_n d\Gamma = \int_{\Gamma_0}\frac{\partial I(\Omega_0)}{\partial n}\phi \boldsymbol{v}_n d\Gamma. \quad (17)$$

On the other hand, assume the test function is in a Hilbert space on the entire domain Ω_t (i.e., $\phi\in H_0^1(\Omega_t)$). The weak form of the Laplace's equation is

$$\int_{\Omega_t}\Delta u(\Omega_t)\phi \, d\Omega = \int_{\Omega_t}\nabla u(\Omega_t)\cdot\nabla\phi \, d\Omega = 0$$

Taking the Fréchet derivative of the first domain integral of this expression (using (7)) yields

$$\int_{\Omega_0}\Delta u'(\Omega_0;\boldsymbol{v})\phi \, d\Omega + \int_{\Gamma_0}\Delta u(\Omega_0;\boldsymbol{v})\phi \boldsymbol{v}_n d\Gamma = \int_{\Omega_0}\Delta u'(\Omega_0;\boldsymbol{v})\phi \, d\Omega = 0, \quad (18)$$

where the first equality is because u is a harmonic function (i.e., $\Delta u = 0$). Putting both Equation (17) and Equation (18) together yields a strong form for $u'(\Omega;V)$,

$$\begin{aligned} u'(\Omega_0;\boldsymbol{v}) &= \left[\frac{\partial I(\Omega_0)}{\partial n} - \frac{\partial u(\Omega_0)}{\partial n}\right]\boldsymbol{v}_n, && \text{on } \Gamma_0, \\ \Delta u'(\Omega_0;\boldsymbol{v}) &= 0, && \text{otherwise.}\end{aligned} \quad (19)$$

Next, we simplify the Fréchet derivative (14) that we introduced in §1.4. For the cost functional (12), we rewrite (14) as a linear form of $\boldsymbol{v}_n$ using an adjoint method. In particular, we solve the adjoint problem,

$$\begin{aligned} p(\boldsymbol{x};\Omega_0) &= 0, && \text{on } \Gamma_0, \\ \Delta p(\boldsymbol{x};\Omega_0) &= u(\boldsymbol{x};\Omega_0) - I(\boldsymbol{x}), && \text{otherwise.}\end{aligned} \quad (20)$$

Using this adjoint solution together with integration by parts and Green's formula, we have

$$\begin{aligned}\int_{\Omega_0}(u(\Omega_0)-I)u'(\Omega_0;\boldsymbol{v})d\Omega &= \int_{\Omega_0}\Delta p u'(\Omega_0;\boldsymbol{v})d\Omega \\ &= \int_{\Omega_0}\mathbf{div}\big(\nabla p(\Omega_0)u'(\Omega_0;\boldsymbol{v})\big)d\Omega - \int_{\Omega_0}\nabla p(\Omega_0)\cdot\nabla u'(\Omega_0;\boldsymbol{v})d\Omega \\ &= \int_{\Gamma_0}\frac{\partial}{\partial n}p(\Omega_0)u'(\Omega_0;\boldsymbol{v})d\Gamma - \int_{\Omega_0}\nabla p(\Omega_0)\cdot\nabla u'(\Omega_0;\boldsymbol{v})d\Omega\end{aligned}$$

For the last integration term, using integration by part yields

$$\int_{\Omega_0} \nabla p(\Omega_0) \cdot \nabla u'(\Omega_0; \boldsymbol{v}) d\Omega = \int_{\Omega_0} \mathbf{div}\big(p \nabla u'\big)\, d\Omega - \int_{\Omega_0} p \Delta u' d\Omega = \int_{\Gamma_0} p \frac{\partial u'}{\partial n} d\Omega - \int_{\Omega_0} p \Delta u' d\Omega$$

Because both p and $\Delta u'$ are zero on the boundary Γ_0, this term vanishes. Meanwhile, from the Dirichlet boundary condition of (19), we obtain

$$\int_{\Omega_0} (u(\Omega_0) - I) u'(\Omega_0; \boldsymbol{v}) d\Omega = \int_{\Gamma_0} \frac{\partial}{\partial n} p(\Omega_0) u'(\Omega_0; \boldsymbol{v}) d\Gamma = \int_{\Gamma_0} \frac{\partial p(\Omega_0)}{\partial n} \left[\frac{\partial I(\Omega_0)}{\partial n} - \frac{\partial u(\Omega_0)}{\partial n} \right] \boldsymbol{v}_n d\Gamma. \tag{21}$$

This is a linear form of $\boldsymbol{v}_n$. The last expression here gives the expression $B_R(\boldsymbol{x})$ in Equation (9) of the paper.

2 Extensions

While this paper focuses on the problem of curve placement, our curve optimization algorithm is more versatile than what we have presented. To demonstrate its generality, this section discusses two extensions to which our method can be easily extended.

2.1 Optimization of Outer Boundary

In §4 of the paper, our curve optimization algorithm uses a fixed outer boundary, which forms an unchanged region Ω to define the residual (12). However, this is by no means a limitation of our algorithm. We can easily extend it to optimize the outer boundary together with curves inside. To this end, we define a new residual function as

$$\bar{R}(\Omega; \mathbb{B}) = \frac{R(\Omega, \mathbb{B})}{A_\Omega} = \frac{1}{2A_\Omega} \int_\Omega (u(\boldsymbol{x}) - I(\boldsymbol{x}))^2 \,\mathrm{d}\Omega, \text{ where } A_\Omega = \int_\Omega \mathrm{d}\Omega.$$

Following the definition of Eulerian derivative, we have a reciprocal rule of this derivative:

$$d\tilde{J}(\Omega_0; \boldsymbol{v}) = \lim_{t \downarrow 0} \frac{\tilde{J}(\Omega_t) - \tilde{J}(\Omega_0)}{t} = \lim_{t \downarrow 0} \frac{1}{t} \frac{J(\Omega_t) A(\Omega_0) - J(\Omega_0) A(\Omega_t)}{A(\Omega_t) A(\Omega_0)} = \frac{1}{A(\Omega_0)^2} \left[dJ(\Omega_0; \boldsymbol{v}) A(\Omega_0) - dA(\Omega_0; \boldsymbol{v}) J(\Omega_0) \right]$$

According to (7), the Fréchet derivative of $A(\Omega_t)$ at Ω_0 is simply $\int_{\Gamma_0} \boldsymbol{v}_n d\Gamma$. Consequently, the Fréchet derivative of $\tilde{J}(\Omega_0)$ is

$$d\tilde{R}(\Omega_0; \boldsymbol{v}) = \frac{1}{A(\Omega_0)} \int_{\Omega_0} (u(\Omega_0) - I) u'(\Omega_0; \boldsymbol{v}) d\Omega - \frac{R(\Omega_0)}{A^2(\Omega_0)} \int_{\Gamma_0} \boldsymbol{v}_n d\Gamma. \tag{22}$$

This equation involves the domain integral over Ω_0. Lastly, applying (21) allows us to express the derivative $d\bar{R}(\Omega; \mathbb{B})$ as a linear form of v_n (see derivation in §3 of the supplementary document):

$$\mathrm{d}\bar{R}(\Omega; \mathbb{B}) = \frac{1}{A_\Omega} \int_\Gamma \left[\frac{\partial p(\boldsymbol{x})}{\partial n} \left(\frac{\partial I(\boldsymbol{x})}{\partial n} - \frac{\partial u(\boldsymbol{x})}{\partial n} \right) - \frac{R(\Omega; \mathbb{B})}{A_\Omega} \right] v_n \,\mathrm{d}\Gamma. \tag{23}$$

With this formula, we can choose the boundary velocity v_n for minimizing the objective function, and thus optimize both the outer boundary and inner curves.

2.2 Regularization of Color Variation on Curves

Diffusion curve images allow the user to edit their color distribution by adjusting color values define along curves. From this point of view, complex color variation along curves can degenerate the user editability. To this end, we add one more regularizer in the objective function (12) of the paper:

$$\hat{R}(\Omega; \mathbb{B}) = \frac{1}{2} \int_\Omega (u(\boldsymbol{x}) - I(\boldsymbol{x}))^2 \,\mathrm{d}\Omega + \alpha \int_{\mathbb{B}} \mathrm{d}\Gamma + \frac{\beta}{2} \int_{\mathbb{B}} (\nabla u(\boldsymbol{x}) \cdot \boldsymbol{t}(\boldsymbol{x}))^2 \,\mathrm{d}\Gamma, \tag{24}$$

where $\boldsymbol{t}$ is the tangential direction pointing along the curve at $\boldsymbol{x}$. The Fréchet derivative of the first term has been presented in (14). The Fréchet derivative of the second term has been introduced above and given in Equation (11) of the paper. In the rest of this section, we focus on the Fréchet derivative of the last term. We denote the last term above as $\hat{L}(\mathbb{B}_t) = \int_{\mathbb{B}_t} (\nabla u(\boldsymbol{x}) \cdot \boldsymbol{t}(\boldsymbol{x}))^2 d\Gamma$.

According to (11), we write the following Fréchet derivative of $\hat{L}(\mathbb{B}_t)$ at the initial curve $\mathbb{B}_0$:

$$
\begin{aligned}
d\hat{L}(\mathbb{B}_0;\boldsymbol{v}) &= 2\int_{\mathbb{B}_0} (\nabla u(\boldsymbol{x}) \cdot \boldsymbol{t}(\boldsymbol{x}))(\nabla u(\boldsymbol{x}) \cdot \boldsymbol{t}(\boldsymbol{x}))' d\Gamma + \int_{\mathbb{B}_0} (\nabla u(\boldsymbol{x}) \cdot \boldsymbol{t}(\boldsymbol{x}))^2 \kappa \boldsymbol{v}_n d\Gamma \\
&= 2\int_{\mathbb{B}_0} (\nabla u(\boldsymbol{x}) \cdot \boldsymbol{t}(\boldsymbol{x}))\left[(\nabla u(\boldsymbol{x}))' \cdot \boldsymbol{t}(\boldsymbol{x}) + \nabla u(\boldsymbol{x}) \cdot \boldsymbol{t}'(\boldsymbol{x})\right] d\Gamma + \int_{\mathbb{B}_0} (\nabla u(\boldsymbol{x}) \cdot \boldsymbol{t}(\boldsymbol{x}))^2 \kappa \boldsymbol{v}_n d\Gamma.
\end{aligned} \tag{25}
$$

At this point, the last term is already a linear form of $\boldsymbol{v}_n$. The first term involves both $\nabla u'(\boldsymbol{x})$ and $\boldsymbol{t}'(\boldsymbol{x})$. For the shape derivative of $\boldsymbol{t}'(\boldsymbol{x})$, we first follow the definition in (9) and obtain

$$
\boldsymbol{t}'(\boldsymbol{x}) = \dot{\boldsymbol{t}}(\boldsymbol{x}) - \nabla_\Gamma \boldsymbol{t}(\boldsymbol{x}) \cdot \boldsymbol{v}.
$$

From the derivation above, we know only the normal velocity is involved in the Fréchet derivative of domain-related integrals. Thus if we set $\boldsymbol{v} = \boldsymbol{v}_n \boldsymbol{n}$, then the last term $\nabla_\Gamma \boldsymbol{t}(\boldsymbol{x}) \cdot \boldsymbol{v}$ vanishes, because $\nabla_\Gamma \boldsymbol{t}(\boldsymbol{x})$, which is a matrix, has every column vector tangential to the curve due to the definition of tangential gradient (10). So we obtain

$$
\boldsymbol{t}'(\boldsymbol{x}) = \dot{\boldsymbol{t}}(\boldsymbol{x}) = \partial_t \boldsymbol{v}_n - \boldsymbol{t}\boldsymbol{t}^T \partial_t \boldsymbol{v}_n.
$$

Similarly, $(\nabla u(\boldsymbol{x}))'$ can be expressed as

$$
(\nabla u(\boldsymbol{x}))' = J[u]\boldsymbol{v}_n
$$

Putting these parts together, we obtain a boundary integral as a linear form of $\boldsymbol{v}_n$. Thereby, the cost functional (24) can be optimized using our algorithm.

3 Removal of Line Segments

In §5.4 of the paper, we mark a line segment as unnecessary when the absolute value of

$$
d_{\boldsymbol{n}}(\boldsymbol{x}) = \frac{\partial u(\boldsymbol{x})}{\partial n_l} + \frac{\partial u(\boldsymbol{x})}{\partial n_r}, \quad \boldsymbol{x} \in \mathbb{B}, \tag{26}
$$

is small. To see the reason mathematically, we first express the solution u of the Laplace's equation, (1) in the paper, as a boundary integral, similar to the formula used in [SXD$^+$12, IKCM13, STZ14]:

$$
u(\boldsymbol{x}) = -\int_\Gamma \left[d_{\boldsymbol{n}}(\boldsymbol{y}) G(\boldsymbol{x};\boldsymbol{y}) - c(\boldsymbol{y}) \frac{\partial G(\boldsymbol{x};\boldsymbol{y})}{\partial n(\boldsymbol{y})} \right] \mathrm{d}\Gamma(\boldsymbol{y}), \tag{27}
$$

where $G(\boldsymbol{x},\boldsymbol{y})$ is the Laplace Green's function; $c(\boldsymbol{y}) = C_l(\boldsymbol{y}) - C_r(\boldsymbol{y})$; and $d_{\boldsymbol{n}}$ follows the definition in (26). On a curve segment in a continuously colored region, $c(\boldsymbol{y})$ always vanishes because of the boundary condition

$$
u(\boldsymbol{x}) = I(\boldsymbol{x}), \;\; \forall \boldsymbol{x} \in \mathbb{B} \cup \partial\Omega, \tag{28}
$$

we used for solving the Laplace's equation. If $d_{\boldsymbol{n}}(\boldsymbol{y})$ is sufficiently close to zero, then the contribution of that curve segment in the boundary integral (27) is negligible, and thus we can safely remove that segment.

4 Second-Order FEM Solve of Poisson's Equation

We use the standard second-order FEM to solve the Laplace's and Poisson's equations arose in the paper. Subjecting to Dirichlet boundary conditions, the equation we solve is

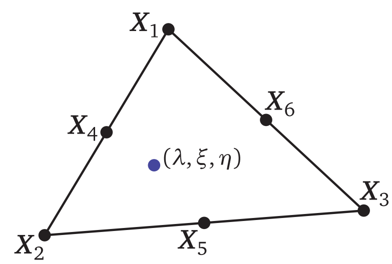
$$
\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), \;\; \boldsymbol{x} \in \Omega. \tag{29}
$$

When we solve the Poisson's equation (20), $f(\boldsymbol{x}) = u(\boldsymbol{x};\Omega_0) - I(\boldsymbol{x})$. When we solve the Laplace's equation (15), $f(\boldsymbol{x}) = 0$. In our FEM implementation, we discretize the domain Ω using a triangle mesh, and construct a linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$. There are plenty of literatures about FEM theory and implementation [ZM71, Wri08]. In this section, we only present necessary formulas for constructing the linear system A in our implementation.

First, the weak form of a Poisson's equation (29) is

$$-\int_{\Omega} \nabla u \cdot \nabla v dS = \int_{\Omega} f v dS. \tag{30}$$

For the Laplace's equation, the right hand side simply vanishes. Inside a triangle of the mesh that discretizes the domain Ω, the solution $u(\boldsymbol{x})$ is approximated using 6 second-order finite element basis $N_I, I = 1...6$:



$$u(\lambda, \xi) = \sum_{I=1}^{6} U_I N_I(\lambda, \xi).$$

where

$$\begin{aligned} N_1 &= \lambda(2\lambda - 1), & N_4 &= 4\xi\lambda \\ N_2 &= \xi(2\xi - 1), & N_5 &= 4\xi\eta \\ N_3 &= \eta(2\eta - 1), & N_6 &= 4\eta\lambda, \end{aligned}$$

Here λ, ξ and η are the barycentric coordinates of $\boldsymbol{x}$ on the triangle. Because $\lambda + \eta + \xi = 1$. We express N_I as a function of λ and ξ, because η is uniquely determined once λ and ξ are known. Similarly the position $\boldsymbol{x}$ parameterized by λ and ξ is determined by the interpolation in a triangle as

$$\boldsymbol{x}(\lambda, \xi) = \sum_{I=1}^{6} \boldsymbol{X}_I N_I(\lambda, \xi).$$

where $\boldsymbol{X}_I$ are the nodal positions of a triangle.

The Jacobian of basis functions N_I is

$$[J_N(\lambda, \xi)] = \begin{bmatrix} \frac{\partial N_1}{\partial \lambda} & \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_2}{\partial \lambda} & \frac{\partial N_2}{\partial \xi} \\ \cdots & \\ \frac{\partial N_6}{\partial \lambda} & \frac{\partial N_6}{\partial \xi} \end{bmatrix},$$

And the Jacobian of $\boldsymbol{x}(\lambda, \xi)$ is

$$[J_e(\lambda, \xi)] = \begin{bmatrix} \frac{\partial x}{\partial \lambda} & \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \lambda} & \frac{\partial y}{\partial \xi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 & \ldots & \boldsymbol{X}_6 \end{bmatrix} [J_N(\lambda, \xi)].$$

Notice in each triangle the nodal positions are fixed, and in our setting, $\boldsymbol{X}_4$, $\boldsymbol{X}_5$, and $\boldsymbol{X}_6$ are respectively the center point of the three triangle edges, that is, $\boldsymbol{X}_4 = \frac{\boldsymbol{X}_1 + \boldsymbol{X}_2}{2}$, $\boldsymbol{X}_5 = \frac{\boldsymbol{X}_2 + \boldsymbol{X}_3}{2}$, and $\boldsymbol{X}_6 = \frac{\boldsymbol{X}_3 + \boldsymbol{X}_1}{2}$. In this case, we obtain a simple form of J_e:

$$J_e = \begin{bmatrix} \boldsymbol{X}_0 - \boldsymbol{X}_2 & \boldsymbol{X}_1 - \boldsymbol{X}_2 \end{bmatrix}.$$

In addition, we have the relationship

$$\begin{bmatrix} \frac{\partial N_1}{\partial \lambda} & \frac{\partial N_2}{\partial \lambda} & \cdots & \frac{\partial N_6}{\partial \lambda} \\ \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \cdots & \frac{\partial N_6}{\partial \xi} \end{bmatrix} = [J_e(\lambda, \xi)]^T \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_6}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_6}{\partial y} \end{bmatrix}.$$

Now we discretize the weak form (30) to obtain the finite element linear system. In particular, for every triangle Δ, the weak form integral is written as

$$\int_{\Delta} \sum_{i=1}^{6} U_i \nabla N_i(\lambda, \xi) \cdot \nabla N_j(\lambda, \xi) \det J_e(\lambda, \xi) d\lambda d\xi = \int_{\Delta} f N_j(\lambda, \xi) \det J_e(\lambda, \xi) d\lambda d\xi, \tag{31}$$

where $\nabla N_i \cdot \nabla N_j$ is computed using

$$\nabla N_i(\lambda,\xi)\cdot\nabla N_j(\lambda,\xi) = \begin{bmatrix} \frac{\partial N_i}{\partial \lambda} \\ \frac{\partial N_i}{\partial \xi} \end{bmatrix}^T [J_e(\lambda,\xi)]^{-1}[J_e(\lambda,\xi)]^{-T} \begin{bmatrix} \frac{\partial N_j}{\partial \lambda} \\ \frac{\partial N_j}{\partial \xi} \end{bmatrix}.$$

In this discretization, the unknowns are discretized value U_I at the triangle mesh's nodal positions. The equation (31) is a linear system of U_I that we need to solve.

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